Entanglement in the Bogoliubov vacuum

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We analyze the entanglement properties of the Bogoliubov vacuum, which is obtained as a second order approximation to the ground state of an interacting Bose–Einstein condensate. We work on one and two dimensional lattices and study the entanglement between two groups of lattice sites as a function of the geometry of the configuration and the strength of the interactions. As our measure of entanglement we use the logarithmic negativity, supplemented by an algorithmic check [1] for bound entanglement where appropriate. The short-range entanglement is found to grow approximately linearly with the group sizes and to be favored by strong interactions. Conversely, long range entanglement is favored by relatively weak interactions. Working with periodic boundary conditions we find some surprising finite size effects for the very long range entanglement. No examples of bound entanglement is found.

I. INTRODUCTION

In the recent years there has been a considerable interest in studies of entanglement in quantum distributed systems. This is a newly developing interdisciplinary field in which quantum information theory meets atomic and molecular physics, quantum optics, condensed matter physics, and quantum statistical physics. There are several areas in which the role of entanglement and quantum information in distributed systems may be studied. The first motivation has come from the studies of quantum macroscopic and mesoscopic phenomena in atomic physics such as Bose-Einstein condensation (BEC) [2, 3, 4]. The group of K. Burnett was perhaps the first to study squeezing and entanglement of quasi-particle excitations in trapped Bose condensates, as well as characterization of dynamical quantum states of a zero temperature BEC [5, 6, 7]. In fact, in multicomponent BEC the non-linear interactions can lead to a squeezing of the collective atomic spin, opening thus a possibility of applications of BEC for precise frequency measurements. This line of research was initiated by Cirac and Zoller and their collaborators [8, 9].

The new impulse to study entanglement in quantum statistical systems has come from the papers of Amico *et al.* [10] and Nielsen [11] and his collaborators, who have considered scaling properties of (short range) entanglement close to a quantum phase transition. Following up on these studies, various spin chain models were considered by a number of authors [12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23]. Perhaps the most interesting result obtained so far in this context concerns the approach of the Garching group, who has not looked only at two parts of the distributed systems by tracing out the rest. On the contrary, Cirac and his colleagues considered localized entanglement of the two parts by performing optimal local measurement on the rest of the system [24]. These authors were able to show that there exists an entanglement length diverging at quantum critical points [25].

Somewhat independently O'Connor and Wootters has introduced general studies of spin chains and rings looking for optimal conditions for entanglement. The optimization has been relate to certain nearest-neighbor Hamiltonian; first in an approximate sense [26, 27, 29], but later exactly [28].

Apart from the spin systems, harmonic chains and rings have been studied [62]. As we will describe below, powerful theoretical tools are available for these systems. An important advantage over spin systems is that entanglement can be studied also between two subsystems containing many sites. In Ref. [30], Audenaert *et al.* used this fact to consider several different subsystems on one-dimensional rings where the coupling between neighboring oscillators is only through position operators (i.e. can be interpreted as "springs"). A general lesson learnt was that the ground state entanglement between two subsystems decreases rapidly with their mutual separation. Increasing the size of the two subsystems in general increases their entanglement, but if the "contact" region between the subsystems is kept fixed, a finite limiting value is eventually reached.

The reason why harmonic chains are relatively easy to treat theoretically is that in a grand canonical description, the ground state as well as the thermal states belong to the category of Gaussian states. Gaussian states are very nice for investigations from a point of view of entanglement properties, since they are completely characterized by their first and second order correlations. This simplification as compared to general infinite dimensional systems has allowed a lot of results to be derived. First of all it has been shown that if Alice and Bob have one harmonic oscillator mode each and share a Gaussian state, the state is entangled if and only if its partial transpose is not positive [31, 32]. If Alice has one mode and Bob many, the same conclusion holds. If, however, both parties have

more than two modes each, they may share an entangled state which has nonetheless a positive partial transpose. For finite dimensionsal systems the existence of such states was demonstrated by the Horodeckis [33], who have shown that these states cannot be used for any entanglement distillation procedures of the kind introduced by Bennett *et al.* [34]. The corresponding distillability problem for Gaussian states was solved with identical result: it was shown that Gaussian states are distillable if and only if their partial transpose is not positive [35]. A necessary and sufficient entanglement criterion for Gaussian states of two parties was found soon after [1]. All these findings allowed for remarkable results concerning classification of Gaussian operations [36], and in particular the proof of the fact that Gaussian states cannot be distilled using Gaussian operations.

As exemplified by this paper, the abovementioned results also have big practical importance, because the Gaussian states of photons and atoms are in many cases the ones that are easily accessible experimentally [37, 38, 39, 40, 41, 42, 43]. In particular, we will here use some of the machinery for Gaussian states to investigate the naturally occurring entanglement in harmonic chains of a very specific kind: The ones that appear in the studies of atoms in optical lattices. Ultra-cold bosonic atoms in optical lattices can undergo a super-fluid to Mott insulator transition, predicted in Ref. [44] (see also [45]), and observed by Greiner *et al.* [46]. The Mott insulator state with its regular filling is considered as an ideal initial state for quantum information processing [47, 48], and this has made these states the subject of intensive investigations in the recent years. We will here instead consider the super-fluid state of the lattice bosonic gas in the intermediate regime where interactions are present, but not completely dominant. Then fluctuations and excitations are described by the Bogoliubov-de Gennes equations [49], i.e. are formally just a system of coupled harmonic oscillators. We therefore find a similar setting as in Ref. [30] albeit with a different form of the coupling between the oscillators. As we consider also two-dimensional lattices, we are faced with an even wider choice of subsystems. We have chosen to focus on subsystems that each consist of a string of contiguous sites and to vary the size, the separation, and (in the two-dimensional case) the relative orientation of the two strings. Finally, we also vary the one remaining physical parameter, namely the ratio between the energy associated with tunneling between sites and the mean-field interaction energy.

Let us briefly summarize our results. As the separation between the two subsystems grows, their entanglement decreases and eventually disappears entirely. This is very similar to the results for the "spring"-chains of Ref. [30]. Using periodic boundary conditions a peculiar finite size effect occurs and the entanglement does not only depend on the distance between the closest parts of the two subsystems, even when this is very well-defined. For the one-dimensional strings we consider, the dependence of the entanglement on their (equal) length is essentially linear up to moderate separations. This holds even when the relative orientation of the strings is changed. The dependence on the interaction strength is such that the entanglement at small separations is increased for increased interactions while the entanglement at large separations is decreased. For a given separation, there is thus an optimal ratio of tunneling and interaction energy with respect to maximizing the entanglement. Somewhat surprisingly, we find no examples of bound entanglement.

The rest of the paper is organized in the following way. First we give some necessary background of quantum information theory in Sec. II. In Sec. III we then calculate the relevant correlations of the system. In Sec. IV we apply the theory to this input and we present and analyze our results. Finally, in Sec. V we discuss our findings and their relations to similar work.

II.   GAUSSIAN STATES AND ENTANGLEMENT

In general, the study of entanglement in continuous variable systems is quite demanding. The important subset Gaussian states is however much easier to treat and for these states the entanglement properties are fairly well understood [1, 36]. The Gaussian states are well known from the field of quantum optics, since important classes like thermal states, coherent states, and squeezed states are all Gaussian. In the field of atom optics, Gaussian states also appear, albeit slightly less naturally since one tends to prefer a description in terms of Fock states, i.e. states with a definite particle number, when dealing with massive particles. This point is discussed further in Sec. V below. For the moment, however, let us just assume that we have M modes described by "position" and "momentum" operators with canonical commutation relations

$$\left[x_\nu, p_{\nu'}\right] = i\delta_{\nu\nu'} \quad , \quad \left[x_\nu, x_{\nu'}\right] = \left[p_\nu, p_{\nu'}\right] = 0. \tag{1}$$

To simplify notation, it is useful to refer to x_ν and p_ν as $r_{2\nu}$ and $r_{2\nu+1}$, respectively. Then Eq. (1) can be written as

$$\left[r_\nu, r_{\nu'}\right] = iJ_{\nu\nu'}, \tag{2}$$

where J is the so-called *symplectic matrix*

$$J = \bigoplus_\nu \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}_\nu. \tag{3}$$

Gaussian states can now be defined as states for which the Wigner characteristic function [50] is Gaussian:

$$\chi_W(\boldsymbol{\xi}) = \left\langle \exp\left(i\boldsymbol{\xi}^{\mathrm{T}}\boldsymbol{r}\right)\right\rangle = \exp\left(i\boldsymbol{\xi}^{\mathrm{T}}\boldsymbol{d} + \frac{1}{4}\boldsymbol{r}^{\mathrm{T}}\gamma\boldsymbol{r}\right). \tag{4}$$

In Eq. (4), $\boldsymbol{d} = \langle\boldsymbol{r}\rangle$ is the average displacement while

$$\gamma_{\nu\nu'} = 2\Re\langle(r_\nu - d_\nu)(r_{\nu'} - d_{\nu'})\rangle \tag{5}$$

is the covariance matrix. Note that a Gaussian state is fully determined by specifying $\boldsymbol{d}$ and γ. In fact, since displacements can always be removed by local operations, only γ is important for the entanglement properties of a state and in the following subsections we summarize how information about the entanglement is extracted from it.

A. Logarithmic negativity

An important problem in quantum information theory is how to meaningfully quantify the amount of entanglement in a state. Viewing entanglement as a resource for performing tasks forbidden by classical physics, one seeks measures that quantify to which extend a state is useful for a certain task or how many standard resources it would take to create the state.

Here we apply as our measure of entanglement the logarithmic negativity [51], E_N. While this measure is by no means perfect, it does have some very nice properties for our purposes. First of all, the value of E_N provides an upper bound on the efficiency of distillation, i.e., the extraction of maximally entangled states from a larger number of less entangled states. Secondly, E_N is an additive quantity which means that it behaves "naturally" when applied to more than one copy of a state. This facilitates comparisons between systems with different numbers of modes. Thirdly, and quite importantly, the logarithmic negativity is computable, i.e., there is an efficient way to calculate it for the states we are interested in.

Formally, the logarithmic negativity is defined for any bipartite density operator ρ as

$$E_N(\rho) = \log_2 ||\rho^{\mathrm{T_A}}||_1, \tag{6}$$

where $\rho^{\mathrm{T_A}}$ is the partially transposed (w.r.t. Alice) density operator and $||\cdot||_1$ denotes the trace norm. For Gaussian states one finds [30]

$$E_N = -\sum_{\nu=1}^{2M} \min\left\{0, \log_2\left[|\lambda_\nu(iJ\tilde{\gamma})|\right]\right\}, \tag{7}$$

where $\lambda_k(iJ\tilde{\gamma})$ denotes the kth eigenvalue of $iJ\tilde{\gamma}$ and $\tilde{\gamma}$ is the covariance matrix of $\rho^{\mathrm{T_A}}$. On the level of covariance matrices, partial transposition is implemented by reversing all momenta belonging to Alice's subsystem.

B. Checking for bound entanglement

One fundamental drawback of E_N is the fact that a value of zero does not guarantee that the examined state is separable: There exist entangled density matrices with positive partial transpose [33, 52]. To complete the picture and check whether a given state with vanishing E_N is indeed separable, we apply a qualitative test in the form of a non-linear algorithm devised by Giedke *et al.* [1]. The idea of this test is to start from the covariance matrix of the state and then successively create a series of new matrices that all exhibit entanglement if and only if the original state does. At each step, simple sufficient (but not necessary) criteria for separability and for entanglement are applied. Quite remarkably it can be shown that a definitive answer will come out in a finite number of steps.

III. FLUCTUATIONS IN THE NON-CONDENSED MODES

In this section we describe the calculation of the fluctuations in the non-condensed modes for bosonic particles on a lattice. We apply the Bogoliubov approximation which will give a good approximation to the ground state as long as almost all atoms are condensed. The calculations are rather straightforward, but we give them in some detail for completeness.

A. The Bose-Hubbard Hamiltonian

Our calculations start with the well-known Bose-Hubbard Hamiltonian [44, 53]. We will work with d dimensional quadratic lattices and have

$$H_{\text{BH}} = 2dJ \sum_{\vec{j}} a_{\vec{j}}^\dagger a_{\vec{j}} - J \sum_{\langle \vec{j}, \vec{j}' \rangle} \left\{ a_{\vec{j}}^\dagger a_{\vec{j}'} + \text{h.c.} \right\} + \frac{g}{2} \sum_{\vec{j}} a_{\vec{j}}^\dagger a_{\vec{j}}^\dagger a_{\vec{j}} a_{\vec{j}}. \tag{8}$$

The first two terms in Eq. (8) describes the possibility for atoms to hop between neighboring sites. They are the discrete equivalents of the kinetic energy with $2d$ the coordination number (number of nearest neighbors) of the lattice. The last term describes the collisional interaction of atoms occupying the same site. The natural modes of the kinetic energy are plane waves and we can rewrite

$$H_{\text{BH}} = J \sum_{\vec{k}} c_{\vec{k}}^\dagger c_{\vec{k}} \, \epsilon_{\text{kin}}(\vec{k}) + \frac{g}{2N_s} \sum_{\vec{k}_1, \vec{k}_2, \vec{k}_3} c_{\vec{k}_1}^\dagger c_{\vec{k}_2}^\dagger c_{\vec{k}_3} c_{\vec{k}_1 + \vec{k}_2 - \vec{k}_3} \tag{9}$$

with $\epsilon(\vec{k}) = 2 \sum_{\sigma = x, y, \dots} (1 - \cos[2\pi k_\sigma])$ the non-interacting energy associated with wave vector $\vec{k}$. N_s is the total number of sites and the plane wave annihilation operators are defined via

$$c_{\vec{k}} = \frac{1}{\sqrt{N_s}} \sum_{\vec{j}} e^{-i 2\pi \vec{k} \cdot \vec{j}} a_{\vec{j}}. \tag{10}$$

B. The Bogoliubov Approximation

Starting from Eq. (9) we now apply the Bogoliubov approximation, i.e. we assume that one mode, the condensate, is macroscopically populated and develop the Hamiltonian to second order in the fraction of non-condensed particles. Formally, we first make the redefinitions

$$c_0 \; \rightarrow \; e^{-i\mu t / \hbar} \left(c_0 + \sqrt{N} \right) \tag{11}$$

$$c_{\vec{k}} \; \rightarrow \; e^{-i\mu t / \hbar} c_k \quad \text{for} \quad \vec{k} \neq \vec{0} \tag{12}$$

$$H_{\text{BH}} \; \rightarrow \; H_{\text{BH}} - \mu \sum_{\vec{k}} c_{\vec{k}}^\dagger c_{\vec{k}}, \tag{13}$$

where N is the number of particles and $\mu = gn$ with $n = N/N_s$ the density of particles. This takes care of the macroscopic population of the condensate and its resulting time-dependence, i.e. we now work in a frame where in the fully condensed approximation all modes are in their vacuum state and the $\vec{k} = \vec{0}$ mode defines zero energy. When only terms of Eq. (9) with at least one factor of N are kept, we get the following quadratic Hamiltonian:

$$H_{\text{quad}} = H_0 + \sum_{\vec{k} \neq \vec{0}} H_{\vec{k}}, \tag{14}$$

where

$$H_0 \; = \; \frac{1}{2} gn \left(2 c_0^\dagger c_0 + c_0^\dagger c_0^\dagger + c_0 c_0 \right) \tag{15}$$

$$H_{\vec{k}} \; = \; \left(\epsilon(\vec{k}) + gn \right) c_{\vec{k}}^\dagger c_{\vec{k}} + \frac{1}{2} gn \left(c_{\vec{k}}^\dagger c_{-\vec{k}}^\dagger + c_{\vec{k}} c_{-\vec{k}} \right). \tag{16}$$

It is well-known how to diagonalize Eq. (16): the normal modes are squeezed combinations of opposite momenta. For each pair $(\vec{k}, -\vec{k})$ we get two bosonic quasi-particle modes,

$$c_+(\vec{k}) \; = \; \cosh \eta_{\vec{k}} \, c_{\vec{k}} + \sinh \eta_{\vec{k}} \, c_{-\vec{k}}^\dagger \tag{17}$$

$$c_-(\vec{k}) \; = \; \sinh \eta_{\vec{k}} \, c_{\vec{k}}^\dagger + \cosh \eta_{\vec{k}} \, c_{-\vec{k}}, \tag{18}$$

with squeezing strength

$$e^{2\eta_{\vec{k}}} = \sqrt{\frac{\lambda \, \epsilon(\vec{k}) + 2}{\lambda \, \epsilon(\vec{k})}}, \tag{19}$$

where $\lambda = J/gn$ quantifies the relative importance of interactions. Energetically, the modes are degenerate and in the limit of vanishing interaction they correspond to the usual plane waves.

We have taken care of all modes except for the condensate itself, the $\vec{k} = 0$ mode. If one tries to actually calculate the squeezing of the condensate fluctuations it is found to be infinite since $\epsilon(\vec{0}) = 0$. This expresses the fact that the true eigenstates of the system must have a well-defined number of particles and therefore the displaced vacuum assumption of Eq. (11) cannot be stationary [54, 55]. The best we can do at this point is to put in by hand a coherent state for the condensate, i.e. to assume that the state we study is annihilated by c_0 (since we have already subtracted the macroscopic population).

C. Fluctuations in a site basis

We now have a simple description of the system in terms of squeezed momentum eigenstates. Since we are interested in the entanglement between different spatial regions of the lattice, we need to calculate the fluctuations and correlations of operators describing the atomic field at each lattice site. To use the notation of most of quantum information theory literature on Gaussian states, we will use quadrature operators defined as

$$x_{\vec{j}} = \frac{a_{\vec{j}} + a_{\vec{j}}^\dagger}{\sqrt{2}} \quad , \quad p_{\vec{j}} = \frac{a_{\vec{j}} - a_{\vec{j}}^\dagger}{i\sqrt{2}}. \tag{20}$$

For the xx-correlations we find squeezing,

$$\langle x_{\vec{j}} x_{\vec{j}'} \rangle - \langle x_{\vec{j}} \rangle \langle x_{\vec{j}'} \rangle = \frac{1}{2N_s} \left\{ 1 + \sum_{\vec{k} \neq \vec{0}} \cos[2\pi \vec{k} \cdot (\vec{j} - \vec{j}')] e^{-2\eta_{\vec{k}}} \right\}, \tag{21}$$

and for the pp-correlations we find anti-squeezing

$$\langle p_{\vec{j}} p_{\vec{j}'} \rangle - \langle p_{\vec{j}} \rangle \langle p_{\vec{j}'} \rangle = \frac{1}{2N_s} \left\{ 1 + \sum_{\vec{k} \neq \vec{0}} \cos[2\pi \vec{k} \cdot (\vec{j} - \vec{j}')] e^{2\eta_{\vec{k}}} \right\}. \tag{22}$$

In Eqs. (21-22), the "1" is the contribution from the condensate mode, i.e., the part we have put in by hand. There are no xp-correlations.

IV. RESULTS

In this section, we present our results for the logarithmic negativity of bipartite states. We begin with results for 1D lattices in Sec. IV A, then move on to 2D lattices in Sec. IV B.

A. 1D lattices

As we are using periodic boundary conditions, our 1D lattice is in fact a ring. We will define the two subsystems between which we want to study the entanglement as two sets of contiguous sites, see Fig. 1. Because of the overall translational invariance, only the size of the groups, q, and their separation, s, matters.

In Fig. 2 we plot E_N as a function of the separation of the two groups for several different group sizes. We note that E_N is generally an increasing function of the group size and a decreasing function of the separation. Small groups even become separable already at moderate separations: $s > 6$ for $q = 1$ and $s > 12$ for $q = 2$. Note that for $q > 1$ we cannot conclude this from the vanishing of E_N alone, but we have to apply the nonlinear algorithm described in Sec. II B.

In Fig. 2, the number of sites is $N_s = 321$ and the finiteness of this number naturally is important when the separation and/or the group size is comparable to it. In that case, there are two relevant distances from Alice to Bob: in the clockwise direction and in the counterclockwise direction. Remarkably, two separations that individually (i.e. as the short distance on an infinite ring) would give rise to $E_N = 0$ can still result in an entangled state when combined. This is evidenced in the insert of Fig. 2 where the vanishing and the revival of E_N as a function of s for $q = 3$ is plotted on a logarithmic scale. On an infinite ring, we would have $E_N = 0$ for all $s > 11$, but we see that
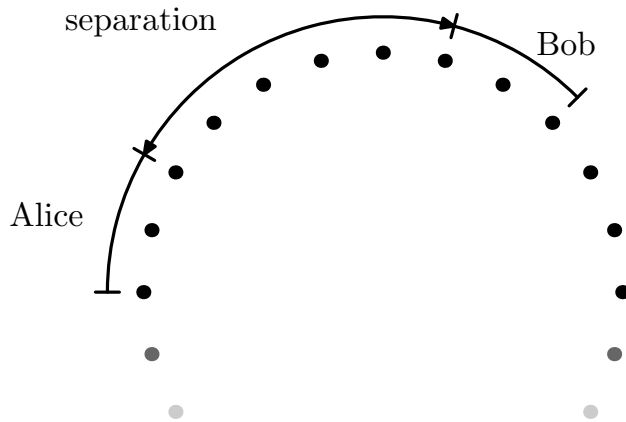
FIG. 1: Definition of the two subsystems. Alice and Bob are both assigned q contiguous sites and the two groups have a separation of s sites (between the extreme sites). Since we are using periodic boundary conditions, i.e. working on a ring, there are in fact two distances between the groups: s and $|N_s - 2q - s| + 2$. We use the convention that s is the smaller one.
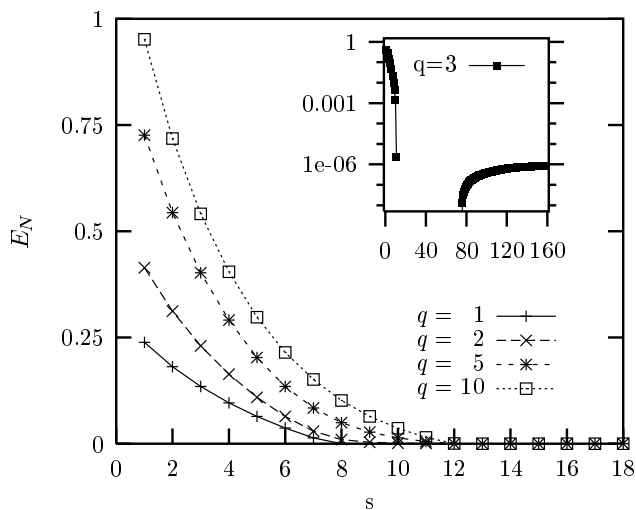

FIG. 2: Logarithmic negativity E_N as a function of group separation s for different group sizes q. In this plot, $N_s = 321$ and $\lambda = J/gn = 20$. When both q and s are much smaller than N_s, E_N is a increasing function of q and a decreasing function of s. The small separation part ($s < 5$) can be reasonably well approximated by an exponential with decay constant ~ 0.3. At larger separations, the decay is faster. In fact, for small group sizes, E_N and thus the distillable entanglement drops strictly to zero at quite moderate separations: $s = 7$ for $q = 1$ and $s = 11$ for $q = 2$. The insert shows a logarithmic plot for $q = 3$. Here the negativity vanishes at $s = 12$, but due to the finiteness of N_s, it reappears at $s = 76$. A similar picture applies for $q = 4$, while for larger qs, finite size effects always keeps E_N non-zero for $N_s = 321$.

at $s = 76$, i.e. at a short separation of 76 and a long of $321 - 2 \cdot 3 - 76 = 138$, E_N again becomes finite. It should of course be noted that the value E_N reaches when the two groups are on opposite sides of the ring is very low ($\sim 10^{-6}$).

Apart from the geometrical aspects regarding the lattice and the definition of Alice's and Bob's subsystems, there is one underlying physical parameter in the problem, namely $\lambda = J/gn$. Small values of this parameter lead to a higher population of atoms with $\vec{k} \neq \vec{0}$ or, equivalently, to more squeezed quasi-particle modes. If we keep n fixed, the Bogoliubov approximation will eventually break down and the system will enter a Mott insulator regime [46]. Note, however, that since we specify only the ratio J/gn, even values that lead to a high absolute number of excited atoms are not *a priori* irrelevant since the Bogoliubov approximation still holds for a high enough n. In Fig. 3 we compare the curves for $q = 3$ and $20 \leq \lambda \leq 100$. At short distances entanglement is clearly favored by a stronger non-linearity (low value of λ), but perhaps a little surprisingly, beyond separations of about 5 sites, stronger non-linearity actually leads to less entanglement. One possible explanation for this phenomenon can be derived from the so-called monogamy of entanglement: Alice's and Bob's subsystems are naturally not only entangled with each other but also with the remaining sites in the system, in particular with the sites in the gap between them. Since the total system is in a pure state, the mixedness of the Alice-Bob system is exactly due to this entanglement. When the short
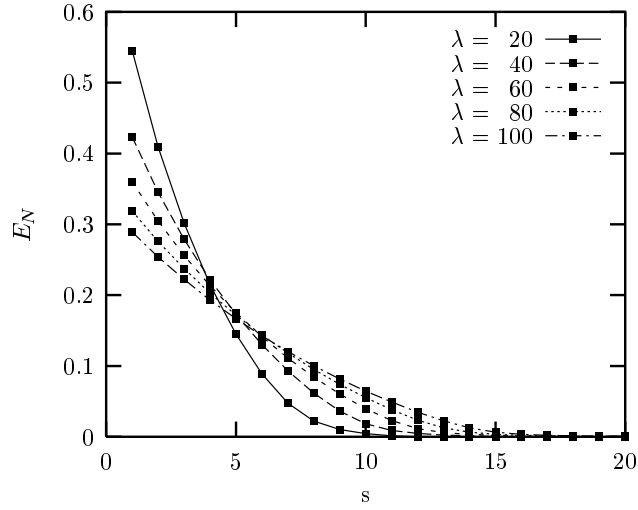
FIG. 3: Dependence of entanglement upon the value of λ. The five curves show E_N as a function of s for $q = 3$ and five different values of λ. At short distances, a higher ratio of hopping to non-linearity leads to less entanglement, while at longer distances the opposite picture applies.
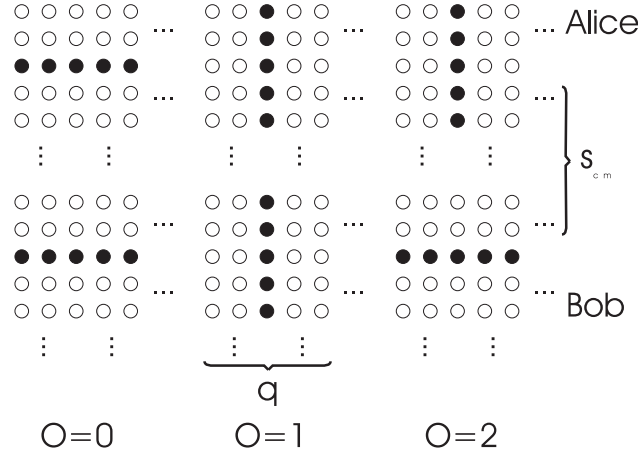

FIG. 4: Assignments of sites to Alice and Bob on the 2D lattice. Note that we again use periodic boundary conditions, i.e. although the lattice is drawn as flat, we work on a torus. The "group size" denoted by q gives the number of sites assigned to both Alice and Bob. The "center-of-mass separation" denoted by s_{cm} gives the distance between central sites in the two group. Finally, the "orientation" denoted by O labels different arrangements of the sites within each group.

distance entanglement then grows due to a stronger non-linearity, the mixedness of a highly delocalized Alice-Bob system must be expected to increase, leaving less room for Alice-Bob entanglement [56]. To test the plausibility of this explanation, we have examined the purity $\mathrm{Tr}\rho^2$ corresponding to Fig. 3. As expected, the purity has a stronger dependence on λ at large separations than at short separations, but the effect is not very pronounced and we cannot rule out entirely different explanations of Fig. 3.

B. 2D lattices

On 2D lattices, we have a wide choice of interesting assignments of sites to Alice and Bob. We have chosen to focus on the ones shown in Fig. 4 and will investigate to which extend the structure within the groups influences the entanglement. To this end, it is natural to first calculate the entanglement between two single sites. In Fig. 5 we therefore show the logarithmic negativity as a function of the separation of the sites. We note that to a good approximation E_N is an isotropic function of the separation and that it decays quickly with increasing distance between the two sites. Compared to the 1D results with the same J/gn ratio, E_N is now a factor of approximately
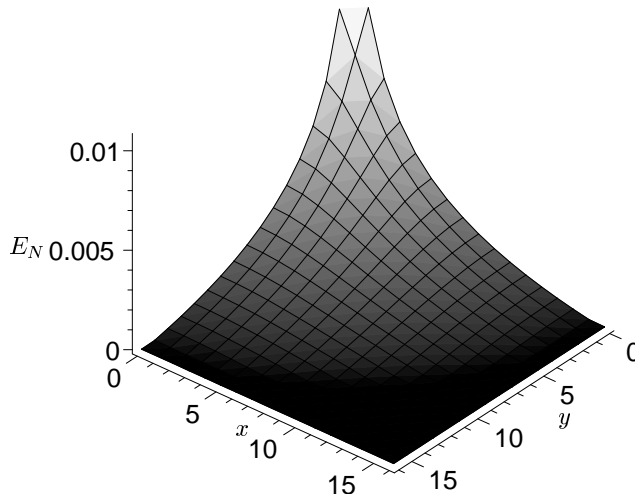
FIG. 5: Entanglement between two single sites as a function of their separation. The full lattice is 241 sites wide in both directions and $\lambda = J/gn = 100$.

10 smaller.

In Fig. 6 we plot our results for the three different "orientations" of the two groups. Instead of plotting directly the logarithmic negativity, we first divide E_N by the number of sites in each group,

$$\tilde{E}_N(q, s_{\mathrm{cm}}) = \frac{1}{q} E_N(q, s_{\mathrm{cm}}), \tag{23}$$

and then subtract the single site result (Fig. 5) evaluated at the "center-of-mass distance" (cf. Fig. 4),

$$\Delta \tilde{E}_N(q, s_{\mathrm{cm}}) = \tilde{E}_N(q, s_{\mathrm{cm}}) - E_N(1, s_{\mathrm{cm}}). \tag{24}$$

This is a sensible rescaling, at least at short distances where $\Delta \tilde{E}_N(q, s_{\mathrm{cm}})$ turns out to be an order of magnitude smaller than $E_N(1, s_{\mathrm{cm}})$. It also almost collapses the curves for group sizes $q = 3$ and $q = 5$ onto each other: for the orientation O=0 to a very high degree, while for O=1 and 2 some residual entanglement remains. That the orientation O=1 where Bob's string is placed as a continuation of Alice's string leads to the highest entanglement at a given s_{cm} is not surprising since this is the orientation with the shortest minimal distance between sites in the two groups.

In 1D we found a remarkable inversion in the dependence of E_N on λ as the separation between groups were increased. A similar effect is present in 2D as can be seen from Fig. 7. There we plot $\tilde{E}_N$ as a function of λ for a number of different group sizes and separations. All the curves show a maximum in the plotted λ range, i.e. in all cases the entanglement is optimized at some finite ratio of hopping to non-linearity. As expected from the successful rescaling in Fig. 6, the optimal λ has only a very weak dependence on the group size. Like in 1D, entanglement between groups at large distances has a larger optimal λ than entanglement at short distances.

V. DISCUSSION

First of all, we should comment on the symmetry breaking Bogoliubov approach that we have applied, i.e. on the procedure of putting in "by hand" a coherent state for the condensate mode. It is well known that this approach leads to identical predictions for excitation frequencies etc. as the symmetry preserving approach that does not assume superpositions of different total numbers of particles [55]. A simple connection between the two can be made by
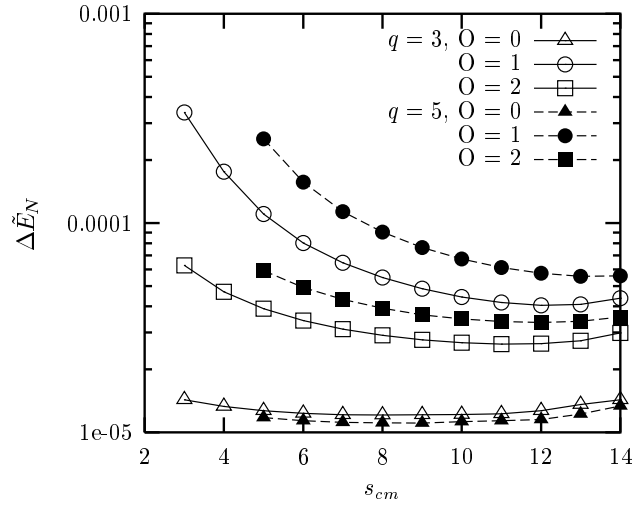
FIG. 6: Results for a 2D lattice with $\lambda = 100$ and a size of 241 sites in each dimension. Group sizes are $q = 3$ (open symbols) and $q = 5$ (filled symbols) and the three pairs of curves correspond to the three orientations described in Fig. 4. The rescaled quantity $\Delta \tilde{E}_N$ is an order of magnitude smaller than the single site logarithmic negativity at separations $s_{\mathrm{cm}} < 11$ indicating that the logarithmic negativity is approximately proportional to q and that s_{cm} is a good measure of the distance between the groups.
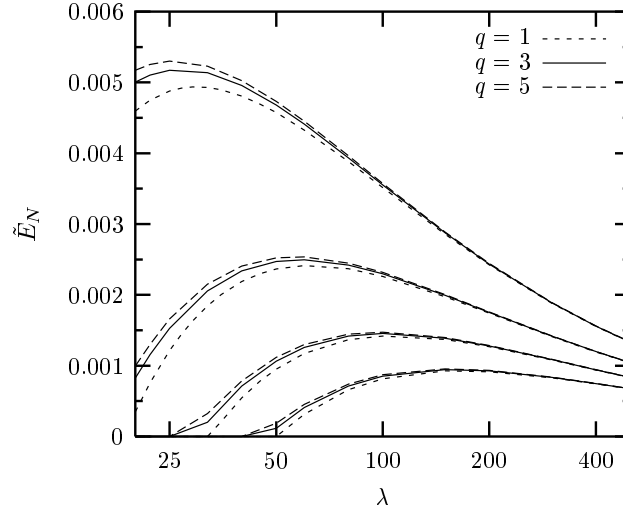

FIG. 7: Dependence of $\tilde{E}_N$ upon λ for the orientation O=0 on a 2D lattice (241 sites in each dimension). The dotted curves are single site results, $q = 1$, the full curves are for groups of $q = 3$ sites each, and the dashed curves are for $q = 5$. Results for 4 different separations are shown: $s_{\mathrm{cm}} = 5, 7, 9$ and 11. The three uppermost curves are for $s_{\mathrm{cm}} = 5$, the next three for $s_{\mathrm{cm}} = 7$ and so on. In the displayed λ range, all the curves show a maximum, i.e. E_N is not a monotone function of λ.

averaging over the phase of the coherent state leaving one with a Poissonian mixture of number states. Instead, our results should be seen as describing the entanglement existing between Alice and Bob when they share knowledge of the overall phase of the condensate. Of course, this phase cannot be regarded as a completely classical piece of information as it can only be defined relative to some reference condensate [57]. For recent investigations regarding the operational implications of *super selection rules* see e.g. Ref. [58] and references therein.

Turning now to our results, the success of the rescaling (Eq. (23)) displayed in Fig. 6 and Fig. 7 is quite remarkable. For the configurations we have considered, the dominant behavior of the logarithmic negativity can be understood in terms the single site result evaluated at s_{cm}: it should simply be scaled with the number of sites in each group. This scaling is exactly what one would expect from the additivity of E_N if the sites in the two groups paired up in a natural way, but such a pairing is far from obvious when looking at Fig. 4. It is an interesting question for future studies to understand more precisely why this scaling applies. One hint can maybe be drawn from recent studies of the asymptotic behavior of "spring"-chains [59].

Another interesting conclusion we can draw is that a high degree of squeezing (strong non-linearity) in the system does not necessarily increase the entanglement: On the contrary, it will eventually tend to decrease long distance entanglement. Similar results have recently been found in related few site systems, both for the ground state [60] and in the entanglement dynamics [61]. It is not clear whether a useful picture of the long distance behavior can be derived from the short distance one via the principle of monogamy of entanglement, but it would be very interesting to at least study the bounds that exist. Since the Bogoliubov model is both relevant and accessible, we expect our results to be useful in such studies, but perhaps the prospects for gaining insight in this way are even greater in quantum critical systems [10, 11].

As a perspective, experimental measurement of the entanglement or even implementation of quantum information processing in the system would of course be interesting. However, in the formulation given here, access to the full covariance matrix of the Alice-Bob system is assumed and a reformulation in terms of more easily accessible quantities would probably be necessary.

Acknowledgements

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